APPENDIX A: TECHNICAL DETAILS

To prove Theorem 1, the five assumptions made in Section 2 of Kiefer and Wolfowitz (1956) need to be verified. This is done in the following.

ASSUMPTION 1. *It is required that $f(y;\boldsymbol{\theta})$ is a density with respect to a σ-finite measure μ on a Euclidean space of which y is the generic point.*

PROOF. This condition is satisfied since the underlying distribution is discrete. □

Define a metric on the space Θ by setting

$$\delta(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)=\sum_{j=1}^{B}|\arctan\beta_{1,j}-\arctan\beta_{2,j}|+\sum_{j=1}^{B}|\arctan\tau_{1,j}-\arctan\tau_{2,j}|.$$

Following Kiefer and Wolfowitz (1956), the parameter space is compactified by defining $\bar{\Theta}$ to be the completion of Θ by adding all the limits of its Cauchy sequences in the sense of the above metric. Unless otherwise mentioned, all limits involving $\boldsymbol{\theta}$ are understood to be with respect to δ. The Euclidean norm is denoted by $|\cdot|_E$. To verify the next assumption of Kiefer and Wolfowitz (1956), two auxiliary lemmas are introduced.

LEMMA 1. *For sufficiently large β_j and a fixed $y\in\mathbb{N}_0$, we have*

$$\beta_j(A\tau_j)^{\beta_j}\int_{A\tau_j}^{A\tau_{j+1}}t^{y-\beta_j-1}e^{-t}dt<2(A\tau_j)^y e^{-A\tau_j},$$

where $\boldsymbol{\tau}\in\Theta$.

PROOF. Note that $\beta_j(A\tau_j)^{\beta_j}\int_{A\tau_j}^{A\tau_{j+1}}t^{y-\beta_j-1}e^{-t}dt\le\beta_j(A\tau_j)^{\beta_j}\Gamma(y-\beta_j,A\tau_j)$. Thus, by Theorem 2.2 of Borwein and Chan (2009), for a sufficiently large β_j and a fixed $y\in\mathbb{N}_0$,

$$\Gamma(y-\beta_j,A\tau_j)\le\frac{-(A\tau_j)^{y-\beta_j}e^{-A\tau_j}}{y-\beta_j}$$

and consequently $\beta_j(A\tau_j)^{\beta_j}\Gamma(y-\beta_j,A\tau_j)<2(A\tau_j)^y e^{-A\tau_j}$. □

LEMMA 2. *If $|\boldsymbol{\tau}|_E<\infty$, $\lim_{|\boldsymbol{\beta}|_E\to\infty}f(y;\boldsymbol{\theta})$ exists.*

PROOF. Note that if $|\boldsymbol{\beta}|_E \to \infty$, there exists a j such that $\beta_j \to \infty$. We focus on that one particular j. Let $g_j(y;\boldsymbol{\theta}) = \beta_j (A\tau_j)^{\beta_j} \int_{A\tau_j}^{A\tau_{j+1}} t^{y-\beta_j-1} e^{-t} dt$. In order to show that the limit of f exists, we only have to show that the limit of g exists (since it generalizes to any j with $\beta_j \to \infty$). Note that, instead of considering g_j, we look at $h_j(y;\boldsymbol{\theta}) = \log g_j(y;\boldsymbol{\theta})$. We define $h_{j,1}(y;\boldsymbol{\theta}) = \log\{\beta_j (A\tau_j)^{\beta_j}\}$ and $h_{j,2}(y;\boldsymbol{\theta}) = \log \int_{A\tau_j}^{A\tau_{j+1}} t^{y-\beta_j-1} e^{-t} dt$. Then we have

$$\frac{\partial h_j(y;\boldsymbol{\theta})}{\partial \beta_j} = \frac{\partial h_{j,1}(y;\boldsymbol{\theta})}{\partial \beta_j} + \frac{\partial h_{j,2}(y;\boldsymbol{\theta})}{\partial \beta_j}$$

$$= \left\{ \frac{1}{\beta_j} + \log(A\tau_j) \right\} + \left\{ -\frac{\int_{A\tau_j}^{A\tau_{j+1}} t^{y-\beta_j-1} e^{-t} (\log t) dt}{\int_{A\tau_j}^{A\tau_{j+1}} t^{y-\beta_j-1} e^{-t} dt} \right\}$$

$$\frac{\partial^2 h_{j,1}(y;\boldsymbol{\theta})}{\partial \beta_j^2} = -\frac{1}{\beta_j^2}$$

$$\frac{\partial^2 h_{j,2}(y;\boldsymbol{\theta})}{\partial \beta_j^2} = \frac{\int_{A\tau_j}^{A\tau_{j+1}} t^{y-\beta_j-1} e^{-t} (\log t)^2 dt}{\int_{A\tau_j}^{A\tau_{j+1}} t^{y-\beta_j-1} e^{-t} dt} - \left\{ \frac{\int_{A\tau_j}^{A\tau_{j+1}} t^{y-\beta_j-1} e^{-t} (\log t) dt}{\int_{A\tau_j}^{A\tau_{j+1}} t^{y-\beta_j-1} e^{-t} dt} \right\}^2$$

Note that $\partial^2 h_{j,2}(y;\boldsymbol{\theta})/\partial \beta_j^2 = \mathrm{Var}(\log(T))$, where T is a random variable with density

$$r(t) = \frac{t^{y-\beta_j-1} e^{-t}}{\int_{A\tau_j}^{A\tau_{j+1}} s^{y-\beta_j-1} e^{-s} ds}, \qquad A\tau_j < t < A\tau_{j+1}.$$

Thus $\partial^2 h_{j,2}(y;\boldsymbol{\theta})/\partial \beta_j^2 \geq 0$ and $\partial h_{j,2}(y;\boldsymbol{\theta})/\partial \beta_j$ is increasing with respect to β_j. It follows then that $\partial h_{j,2}(y;\boldsymbol{\theta})/\partial \beta_j$ is bounded from above since h_j is bounded from above by Lemma 1. Consequently, $\lim_{\beta_j \to \infty} \partial h_{j,2}(y;\boldsymbol{\theta})/\partial \beta_j$ exists. □

ASSUMPTION 2 (Continuity Assumption). *It is possible to extend the definition of $f(y;\boldsymbol{\theta})$ so that the range of $\boldsymbol{\theta}$ will be $\bar{\Theta}$ and so that, for any $\{\boldsymbol{\theta}_i\}$ and $\boldsymbol{\theta}^*$ in $\bar{\Theta}$, $\boldsymbol{\theta}_i \to \boldsymbol{\theta}^*$ implies*

$$f(y;\boldsymbol{\theta}) \to f(y;\boldsymbol{\theta}^*)$$

except perhaps on a set of y whose probability is 0 according to the probability density $f(y;\boldsymbol{\theta}_0)$. (The exceptional y-set may depend on $\boldsymbol{\theta}^*$ and $f(y;\boldsymbol{\theta}^*)$ need not be a probability density function.)

PROOF. First, $f(y;\boldsymbol{\theta})$ is continuous with respect to $\boldsymbol{\theta} \in \Theta$ and thus f automatically fulfills the above continuity requirement for $\boldsymbol{\theta} \in \Theta$. Define $\partial\Theta = \bar{\Theta}\backslash\Theta$. Now, we will show that we can define $f(y;\boldsymbol{\theta}^*)$, where $\boldsymbol{\theta}^* \in \partial\Theta$, as $\lim_{\boldsymbol{\theta}\to\boldsymbol{\theta}^*} f(y;\boldsymbol{\theta})$. It is thus only required to show the existence of this limit. Notice that $\lim_{\boldsymbol{\theta}\to\boldsymbol{\theta}^*} f(y;\boldsymbol{\theta})$ exists for boundary points $\boldsymbol{\theta} \in \partial\Theta$ with $|\boldsymbol{\theta}|_E \neq \infty$. The remaining case $|\boldsymbol{\theta}|_E = \infty$ can be separated into three subcases: (i) $|\boldsymbol{\beta}|_E = \infty$ and $|\boldsymbol{\tau}|_E < \infty$, (ii) $|\boldsymbol{\beta}|_E < \infty$ and $|\boldsymbol{\tau}|_E = \infty$, and (iii) $|\boldsymbol{\beta}|_E = \infty$ and $|\boldsymbol{\tau}|_E = \infty$.

1. Suppose $|\boldsymbol{\beta}|_E = \infty$ and $|\boldsymbol{\tau}|_E < \infty$. From Lemma 2, $\lim_{|\boldsymbol{\beta}|_E\to\infty} f(y;\boldsymbol{\theta})$ exists.
2. Suppose $|\boldsymbol{\beta}|_E < \infty$ and $|\boldsymbol{\tau}|_E = \infty$. This implies that there exists at least one j such that $\tau_j = \infty$. Here, we have
$$0 \leq a^{\beta_j} \int_{Aa}^{Ab} t^{y-\beta_j-1} e^{-t} dt \leq a^{\beta_j} \int_{Aa}^{\infty} t^{y-\beta_j-1} e^{-t} dt,$$
where $0 < a < b$. Taking the limit on the right-hand side, using the l'Hospital rule, it follows that
$$\lim_{a\to\infty} \frac{\int_{Aa}^{\infty} t^{y-\beta_j-1} e^{-t} dt}{a^{-\beta_j}} = \lim_{a\to\infty} \frac{A^{y-\beta_j-1} \tau_j^y e^{-Aa}}{\beta_j} = 0.$$
Since $0 \leq (c/a)^{\beta_j-1} \leq 1$ for all $0 < c < a$, $\lim_{|\boldsymbol{\tau}|_E\to\infty} f(y;\boldsymbol{\theta})$ exists.
3. $|\boldsymbol{\beta}|_E = \infty$ and $|\boldsymbol{\tau}|_E = \infty$. The existence of $\lim_{|\boldsymbol{\theta}|_E\to\infty} f(y;\boldsymbol{\theta})$ is basically implied by Lemma 1.

The proof is complete. □

ASSUMPTION 3. *For any $\boldsymbol{\theta}$ in $\bar{\Theta}$ and any $\rho > 0$, $w(y;\boldsymbol{\theta},\rho)$ is a measurable function of y, where*
$$w(y;\boldsymbol{\theta},\rho) = \sup f(y;\boldsymbol{\theta}'),$$
the supremum being taken over all $\boldsymbol{\theta}'$ in $\bar{\Theta}$ for which $\delta(\boldsymbol{\theta},\boldsymbol{\theta}') < \rho$.

PROOF. The statement is implied by the continuity of $f(y;\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta} \in \bar{\Theta}$. □

ASSUMPTION 4 (Identifiability Assumption). *If $\boldsymbol{\theta}$ in $\bar{\Theta}$ is different from $\boldsymbol{\theta}_0$, then, for at least one x,*
$$\int_{-\infty}^{x} f(y|\boldsymbol{\theta}) d\mu \neq \int_{-\infty}^{x} f(y|\boldsymbol{\theta}_0) d\mu,$$
the integral being over those y all of whose components $\leq$ the corresponding of x.

PROOF. In the present case, μ is the counting measure and thus, for all $\boldsymbol{\theta} \in \bar{\Theta}$, if $f(y|\boldsymbol{\theta}) \neq f(y|\boldsymbol{\theta}_0)$ for at least one $y \in \mathbb{N}_0$, it fulfills the above assumption. This is obviously true for $\boldsymbol{\theta} \in \Theta$. Since $\boldsymbol{\theta}_0 \in \Theta$, it is also easy to see that the above is true for $\boldsymbol{\theta} \in \bar{\Theta}$. □

ASSUMPTION 5 (Integrability Assumption). *For any $\boldsymbol{\theta}$ in $\bar{\Theta}$ we have*

$$\lim_{\rho \downarrow 0} \mathbb{E}\left[\log \frac{w(Y; \boldsymbol{\theta}, \rho)}{f(Y; \boldsymbol{\theta}_0)}\right]^+ < \infty,$$

where w is defined in Assumption 3.

PROOF. Since $f(y; \boldsymbol{\theta})$ is continuous and bounded over $\bar{\Theta}$, $\log w(y; \boldsymbol{\theta}, \rho)$ is bounded from above. Now, we want to show that $\mathbb{E}|\log f(Y; \boldsymbol{\theta}_0)| < \infty$. Since $f(y; \boldsymbol{\theta}_0)$ is bounded from above, we only need $\mathbb{E}\{\log(f(Y; \boldsymbol{\theta}_0)\} > -\infty$, which can be shown as follows. Note that, for any $\boldsymbol{\theta} \in \Theta$,

$$\begin{aligned}
\mathbb{E}[\log\{f(Y;\boldsymbol{\theta})\}] &= \mathbb{E}\left[\log\left\{\sum_{j=1}^{B}\left(\frac{\tau_{j-1}}{\tau_j}\right)^{\beta_{j-1}} \frac{\beta_j (A\tau_j)^{\beta_j}}{Y!} \int_{A\tau_j}^{A\tau_{j+1}} t^{Y-\beta_j-1} e^{-t} dt\right\}\right] \\
&\geq \sum_{j=1}^{B}\left[\log\left\{\left(\frac{\tau_{j-1}}{\tau_j}\right)^{\beta_{j-1}} \beta_j (A\tau_j)^{\beta_j}\right\} + \mathbb{E}\left\{\log\left(\frac{\int_{A\tau_j}^{A\tau_{j+1}} t^{Y-\beta_j-1} e^{-t} dt}{Y!}\right)\right\}\right] \\
&\geq \sum_{j=1}^{B} \log\left\{\left(\frac{\tau_{j-1}}{\tau_j}\right)^{\beta_{j-1}} \beta_j (A\tau_j)^{\beta_j}\right\} + \sum_{j=1}^{B} \mathbb{E}\left\{\log\left(\frac{\int_{A\tau_j}^{\infty} t^{Y-\beta_j-1} e^{-t} dt}{Y!}\right)\right\} \\
&= \sum_{j=1}^{B} \log\left\{\left(\frac{\tau_{j-1}}{\tau_j}\right)^{\beta_{j-1}} \beta_j (A\tau_j)^{\beta_j}\right\} + \sum_{j=1}^{B} \mathbb{E}\left[\log\left\{\frac{\Gamma(Y-\beta_j, A\tau_j)}{\Gamma(Y+1)}\right\}\right]
\end{aligned}$$

Here,

$$\frac{\Gamma(Y-\beta_j, A\tau_j)}{\Gamma(Y+1)} = \frac{\Gamma(Y-\beta_j, A\tau_j)}{\Gamma(Y-\beta_j)} \frac{\Gamma(Y-\beta_j)}{\Gamma(Y+1)} = Q(Y-\beta_j, A\tau_j) \frac{\Gamma(Y-\beta_j)}{\Gamma(Y+1)},$$

where Q is the regularized incomplete gamma function. Now, we state the asymptotic expansions of the regularized incomplete gamma function and the ratio of two gamma functions: When $a \to \infty$,

$$\begin{aligned}
Q(a,z) &\propto 1 - \frac{a^{-a-1/2} e^{a-z} z^a}{\sqrt{2\pi}} \left\{1 + O\left(\frac{1}{a}\right)\right\}, \\
\frac{\Gamma(a+b)}{\Gamma(a+c)} &\propto a^{b-c} \left\{1 + O\left(\frac{1}{a}\right)\right\}.
\end{aligned}$$

Applying these asymptotic expansions for large y,

$$\frac{\Gamma(y-\beta_j, A\tau_j)}{\Gamma(y+1)} \propto y^{-\beta_j-1} \left\{1 + O\left(\frac{1}{y}\right)\right\}.$$

Thus, in order to bound $\mathbb{E}[\log\{\Gamma(Y-\beta_j, A\tau_j)/\Gamma(Y+1)\}]$, we only have to bound $\mathbb{E}\{\log(Y)1_{\{Y\geq M\}}\}$ away from ∞ for sufficiently large M. Here, we define $\log 0 \times 0 = 0$. Now, we only have to consider the boundedness of $\sum_{y=M}^{\infty} \log(y)/y^{\beta_j+1}$ for $j = 1, \ldots, B$. It is bounded whenever $\beta_j > 0$, which is fulfilled by any $\boldsymbol{\theta} \in \Theta$. Thus, $\mathbb{E}\{\log(f(Y;\boldsymbol{\theta}_0)\} > -\infty$ since $\boldsymbol{\theta}_0 \in \Theta$. □

The statement of Theorem 1 follows now from Section 2 of Kiefer and Wolfowitz (1956).

Department of Statistics
University of California at Davis
4118 Mathematical Sciences Building
One Shields Avenue
Davis, CA 95616
E-mail: rkwwong@ucdavis.edu
pdbaines@ucdavis.edu
aaue@ucdavis.edu
tcmlee@ucdavis.edu

Harvard-Smithsonian Center for Astrophysics
60 Garden Street
Cambridge, MA 02138
E-mail: kashyap@head.cfa.harvard.edu